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A DIFFERENTIAL MODEL OF FREQUENCY-INDEPENDENT ENERGY DISSIPATION DURING VIBRATIONS†

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A non-linear model which describes the dissipative energy losses which are independent of the frequency of the vibrations but depend considerably on the amplitude is described. The model enables one to take account uniformly of the energy dissipation in structures and materials under steady-state and transient conditions, including wave processes. The parameters of the model are determined on the basis of experimental data for materials with characteristic forms of the dependence of the dissipation on the amplitude. Analytic solutions of the equation of the model and expressions for the absorption coefficient are obtained in a number of important practical cases. In other cases, solutions are obtained by numerical integration. A model problem on the longitudinal vibrations of a homogeneous and composite rod are solved using the Godunov difference scheme. © 2002 Elsevier Science Ltd. All rights reserved.

It is well known that structural damping is practically independent of the frequency but shows a considerable dependence on the amplitude. The dissipation in structural materials, when the stresses are sufficiently large and they are of the order of the endurance limit, has a similar form. The above-mentioned forms of losses usually determine the main dissipation of energy accompanying the vibrations of structures. It is these that will be borne in mind when we are considering the internal dissipation of energy. Sometimes, they can be taken into account by introducing separate hysteresis loop parameters into the calculations [1, 2]. However, such an approach is, strictly speaking, only applicable to steady-state conditions or close to steady-state conditions. As far as structural damping is concerned, its investigation usually reduces to investigating the losses due to dry friction in the contacting surfaces in the individual forms of fixed joints [2, 3].

Linear viscoelastic models are the most convenient in practical calculations. In addition, they are not connected with the actual type of a dynamic process. However, the vibration decrements described by them depend very much on the frequency and are independent of the amplitude. Their domain of applicability is therefore restricted and the problem of developing an adequate and sufficiently universal model to describe the internal dissipation of energy continues to be an urgent problem.

1. DESCRIPTION OF THE MODEL

An analysis of the relations between the stresses and strains in problems with external and internal friction has shown that the existence of a complex $\sigma - \sigma_m$, where σ_m is the stress at the instant the direction of the load changes, is their characteristic feature. We shall assume that the energy dissipation is associated with the relaxation of just $\sigma - \sigma_m$ and not the total stress. We will consider a three-element model, which is structurally a Maxwell model for which a source of the stress σ_m attained at the instant the last change in the value of sign $(\dot{\varepsilon}\sigma)$ occurs (the dot denotes differentiation with respect to t), is included in parallel with the viscous element,

$$\varepsilon = \sigma / E + \varepsilon_v, \quad \sigma = \sigma_m + \sigma_v, \quad \sigma_v = |\mu\dot{\varepsilon}_v|^{1/\alpha} \operatorname{sign} \dot{\varepsilon}_v \tag{1.1}$$

where E is the modulus of elasticity, μ and ε_v are the viscosity coefficient and the strain of the viscous element, σ_v is the stress in the viscous element and t is the time. In the general case, the drag of the viscous element depends non-linearly on $\dot{\varepsilon}_v$. When $\alpha = 1$, in the case of a constant coefficient μ, we have damping with a linear characteristic $\sigma_v = \mu\dot{\varepsilon}_v$.

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For convenience in the subsequent discussion, we will assume that all the stresses are divided by the modulus of elasticity E and we will retain the earlier notation for them. It follows from relations (1.1) that

$$\sigma - \sigma_m = |\mu(\dot{\varepsilon} - \dot{\sigma})|^{1/\alpha} \operatorname{sign}(\dot{\varepsilon} - \dot{\sigma})$$

After raising this to the power α and some reduction, we obtain

$$\dot{\sigma} = \dot{\varepsilon} - \left[|\sigma - \sigma_m|^\alpha \operatorname{sign}(\sigma - \sigma_m)\right] / \mu$$

Suppose $1/\mu = \beta|\dot{\varepsilon}|$. Then, taking account of the fact that sign $(\sigma - \sigma_m) =$ sign $\dot{\varepsilon}$, we have

$$\dot{\sigma} = \left(1 - \beta|\sigma - \sigma_m|^\alpha\right)\dot{\varepsilon} \tag{1.2}$$

The ability simultaneously to fix constant levels of the stresses and strains is a distinctive feature of model (1.2). It can be considered as a Maxwell model with a variable relaxation time, which is connected by means of a specific non-linear relation with the stresses and strains

$$\dot{\sigma} = \dot{\varepsilon} - \sigma/\tau, \quad \tau^{-1} = \beta\dot{\varepsilon}|\sigma - \sigma_m|^\alpha / \sigma \tag{1.3}$$

Maxwell models with a variable relaxation time are used, for example, to describe the high-speed deformation of materials [4].

The action of harmonic stresses $\sigma = A \sin \omega t$ is an important special case. To be specific, we will assume that there are no initial stresses and strains. When $\alpha = 0.5$, 1 and 2, relation (1.2) can be integrated in closed form and we can obtain analytic expressions for the strains

$$\begin{aligned}
&\varepsilon_+ = -\frac{2A}{z^2}\left[\sqrt{s}z + \ln\left(1 - \sqrt{s}z\right)\right] \\
&\varepsilon_- = \frac{2A}{z^2}\left[\ln\left(1 - \sqrt{1 - sz}\right) + \sqrt{1 - sz} - z - \ln(1 - z)\right], \quad \alpha = 0.5 \\
&\varepsilon_+ = -\frac{A}{z}\ln(1 - sz), \quad \varepsilon_- = \frac{A}{z}\{\ln[1 - (1 - s)z] - \ln(1 - z)\}, \quad \alpha = 1 \\
&\varepsilon_+ = \frac{A}{2\sqrt{z}}\ln\frac{1 + s\sqrt{z}}{1 - s\sqrt{z}} \\
&\varepsilon_- = \frac{A}{2\sqrt{z}}\left[\ln\frac{1 + \sqrt{z}}{1 - \sqrt{z}} - \ln\frac{1 + (1 - s)\sqrt{z}}{1 - (1 - s)\sqrt{z}}\right], \quad \alpha = 2 \\
&s = \sin\omega t, \quad T = 2\pi/\omega, \quad z = \beta A^\alpha < 1
\end{aligned} \tag{1.4}$$

where ε_+ corresponds to the loading in the time interval $0 < t \leqslant T/4$, and ε_- corresponds to the unloading in the interval $T/4 < t \leqslant T/2$. In the second half-period, the corresponding $\varepsilon(t)$ relations for the loading and unloading change their signs.

We will adopt the absorption coefficient ψ as a measure of the energy dissipation. This coefficient is defined as the ratio of the energy dissipated during a cycle of vibrations to the maximum potential energy in the cycle [2]. The quantities ψ, corresponding to (1.4), are defined by the relations

$$\begin{aligned}
&\psi = \frac{8}{z^2}[z + \ln(1 - z)] + \frac{8}{3z^4}\left[2(1 - z)^3 - 9(1 - z)^2 + 18(1 - z) - 11 - 6\ln(1 - z)\right], \quad \alpha = 0.5 \\
&\psi = -\frac{4}{z^2}[2z + (2 - z)\ln(1 - z)], \quad \alpha = 1 \\
&\psi = -\frac{2}{z}\left[\ln(1 - z) + \left(1 + \sqrt{z}\right)\ln\left(1 + \sqrt{z}\right) + \left(1 - \sqrt{z}\right)\ln\left(1 - \sqrt{z}\right)\right], \quad \alpha = 2
\end{aligned} \tag{1.5}$$

For other α and $\sigma(t)$, the values of $\varepsilon(t)$ and ψ can be determined by numerical integration of Eq. (1.2). When the loading or unloading is not monotonic, additional "imbedded" hysteresis loops appear in a trajectory in the (σ, ε)-plane. If an internal loop is closed, and the loading and unloading is continued, then σ_m takes a value corresponding to the nearest external loop or to the unclosed trajectory along which further deformation occurs. This algorithm for the change in σ_m is quite simply implemented during the numerical integration procedure.

2. FORMULATION OF THE MODEL PROBLEM

The model for the internal dissipation of energy described above has been used to investigate the longitudinal vibrations of a rod. In this case, the dynamic equations for a straight rod of constant cross-section take the form

$$\sigma' = \dot{U}, \quad \dot{\sigma} = \left(1 - \beta|\sigma - \sigma_m|^{\alpha}\right)U' \tag{2.1}$$

Here U and x are the velocity and coordinate of a section along the axis of the rod and a prime denotes a derivative with respect to x. As above, the stresses are divided by the modulus of elasticity, U is divided by the velocity of sound in the medium, x is divided by the length of the rod and t is divided by the transit time of an acoustic wave along the rod.

The rod is clamped in the section $x = 1$. The dynamic behaviour in the section $x = 0.5$ will be considered as the characteristic behaviour.

System (2.1) was integrated numerically using Godunov's method [5]. In a uniform mesh with cell sides $\Delta x = x_n - x_{n-1}$ and $\Delta t = t^* - t_*$

$$\begin{aligned} &U^* = U_* + (\sigma_n - \sigma_{n-1})\Delta t/\Delta x \\ &\sigma^* = \sigma_* + \left(1 - |\sigma^* - \sigma_m|^{\alpha}\right)b, \quad b = (U_n - U_{n-1})\Delta t/\Delta x \end{aligned} \tag{2.2}$$

The mean integral values on the sides of a computational cell t^* and t_* are labelled using an asterisk as a superscript and subscript, respectively, and the mean integral values on the sides x_{n-1} and x_n are labelled with subscripts $n - 1$ and n.

Newton's method is used to solve the second equation of (2.2)

$$\sigma^{i+1} = \sigma^i - \frac{\sigma^i - \sigma_* - b + \beta b|\sigma^i - \sigma_m|^{\alpha}}{1 + \alpha\beta|b|\,|\sigma^i - \sigma_m|^{\alpha-1}}, \quad \sigma^0 = \sigma_* + b \tag{2.3}$$

where σ^i is the value of σ^* at the i-th iteration. Here, account has been taken of the fact that

$$\operatorname{sign}(\sigma^* - \sigma_m) = \operatorname{sign}\dot{\varepsilon} = \operatorname{sign} b$$

The relations in the characteristics of system (2.1)

$$\sigma_n + jU_n = \left(\sigma + jU - \beta b|\sigma - \sigma_m|^{\alpha}/2\right)_j \tag{2.4}$$

where j is the direction cosine of the inward normal to the side $x = x_n$ of a computational cell, were used to calculate the decay of the discontinuity when $x = x_n$. The subscript $j = -1$ corresponds to a cell in the layer t_*, adjoining $x = x_n$ on the side where $x < x_n$ and the subscript $j = +1$ corresponds to a similar cell on the other side where $x > x$.

It was assumed in the calculations that $\Delta t = \Delta x = 0.001$ and three iterations were carried out using Newton's method. A check on the convergence showed that the error in calculating the amplitudes of the stress did not exceed 0.02% in this case.

3. RESULTS OF NUMERICAL CALCULATIONS

Calculations of the hysteresis loop for different forms of $\sigma(t)$, including those having the form of a "linear sine" and meanders with different rates of increase and decrease of the load, showed that the hysteresis loops defined by Eq. (1.2) and, consequently, also the absorption coefficients, are independent of the frequency. They remained unchanged if, within the limits of each cycle, the loading and unloading are monotonic functions of time. The hysteresis loops for $A = 1$, $\beta = 0.8$ and different values of α are shown on the left of Fig. 1. Because of distortions of the relation $\varepsilon(t)$, both he dimensions as well as the shape of a loop vary when the dissipation parameters α and β vary. The characteristic distortions $\varepsilon(t)$ are seen on the right of Fig. 1 for stresses which vary according to a "linear sine" law. This law is also represented by the dashed curve.

By relations (1.2) and (1.5), the absorption coefficient is a function of α and z. If the stresses are normalized using the different characteristic values of M and M_1, then, for a fixed value of α, it follows from the condition that ψ is invariant that

$$\beta_1 = \beta(M_1 / M)^\alpha \tag{3.1}$$

The stresses will hence forth be normalized using the maximum value. In particular, when they vary harmonically, it will be assumed that $A = 1$. When the second normalized value of M_1 is chosen, the results are recalculated with simple multiplication by M_1/M and they will correspond to the value of β_1 defined by expression (3.1).

The values of α and β for actual materials and structures can be determined by carrying out a non-linear regression using model (1.2) for the experimental $\psi(A)$ relations. These relations are shown in Fig. 2 for a cable (the open circles) and a single wire (the dark circles) when $M = 98$ MPa and, also, for 12×13 steel after tempering (the small crosses) when $M = 343$ MPa [6]. The continuous curves correspond to relations which describe model (1.2) for the following values of the parameters: $\alpha = 0.61$, $\beta = 0.19$ (curve 1), $\alpha = 0.92$, $\beta = 0.04$ (curve 2), and $\alpha = 1.75$, $\beta = 0.094$ (curve 3). We shall henceforth call the $\psi(A)$ relations the energy-dissipation amplitude characteristics. It can be seen that the model reproduces the experimental curves quite well even when there is a tenfold change in the amplitude. At the same time, amplitude characteristics of different types: soft (1), linear (2) and hard (3) are modelled.

The type of $\psi(A)$ relation has a substantial effect on the shape of the wave front. The leading part of the stress wave, which arises in the rod when a constant unit stress is suddenly applied to the end of the rod $x = 0$, is shown in Fig. 3 at the instant $t = 1$. In the case of a small value of α, that is, in the case of a soft amplitude characteristic, the front is concave and has a positive curvature. When $\alpha > 1$, which corresponds to a hard dissipation characteristic, it becomes convex. The parameter β mainly determines the width of the front and its velocity. As β increases, the velocity of the front decreases, it becomes very blurred and broader. After the passage of the front, a constant stress is established behind it.

Decay of the discontinuity is observed at the boundary of segments of the rod with a different level of attenuation and it is more pronounced for segments with soft dissipation characteristics. The decay of the discontinuity as the wave passes from a segment with relatively strong attenuation ($\beta = 0.7$) to a segment with weaker damping ($\beta = 0.5$) is represented by curve *1* in Fig. 4. For the whole rod $\sigma = 0.1$. The decay of the discontinuity is accompanied by an increase in the stresses in the reflected wave and the passing wave. The decay of the discontinuity at the joint of the segments, arranged in the reverse order, represented by curve *2*. In this case, after the front passes across the joint, the stresses

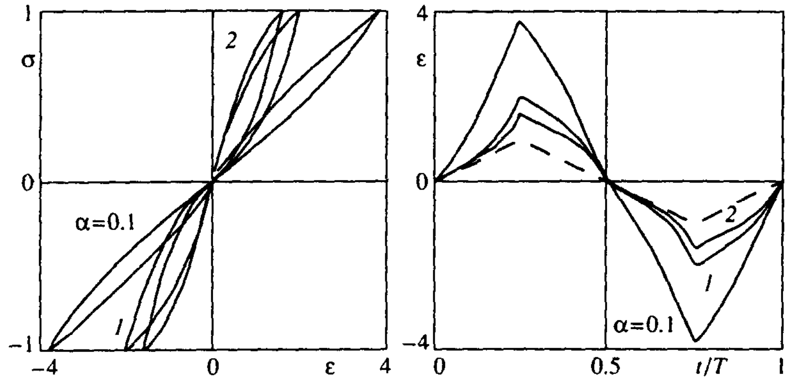

Fig. 1

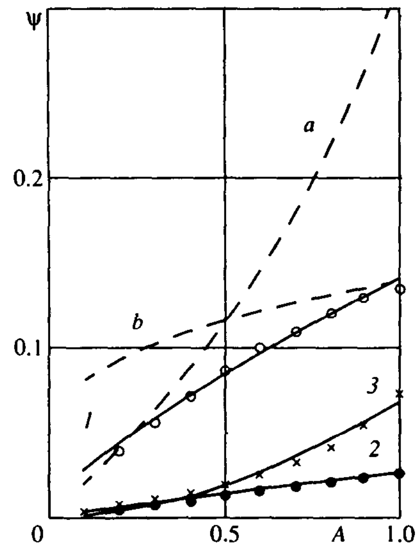

Fig. 2

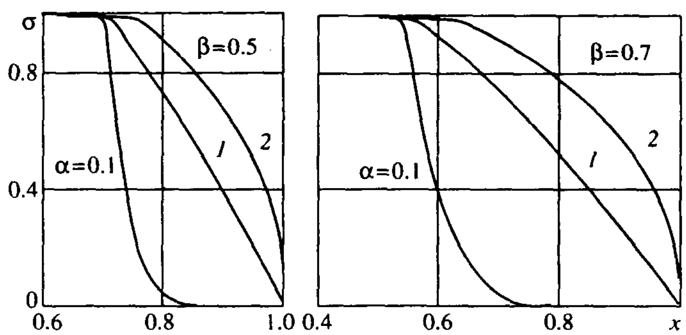

Fig. 3

on both sides of it are less than in a homogeneous rod. The wave in a homogeneous rod when $\beta = 0.5$ is represented by the dashed curve.

The free vibrations of a prestretched rod with initial stresses $\sigma = 1$ in all sections were considered. Its end $x = 0$ was released at the instant $t = 0$. The amplitude envelope when $\alpha = 0.93$ (curve a) and $\alpha = 0.1$ (curve b) are shown on the left of Fig. 5. In both cases it is assumed than $\beta = 0.5$. The corresponding $\psi(A)$ relations are also shown in Fig. 2 by curves a and b. The dissipation characteristic a is soft and b is hard, and their parameters are chosen in such a way that, when $A = 0.5$, the absorption coefficient is the same. In the case of a soft amplitude characteristic, the vibrations initially, for large amplitudes ($t < 200$), decay more slowly and, when the amplitudes become small, they decay more rapidly than in the case of a hard characteristic.

The existence of segments with a different energy dissipation can introduce noticeable irregularities into free vibrations, especially at the initial stage. The free vibrations of a rod, for half of which, $x \leqslant 0.5$, the parameter $\beta = 0.5$ and for the other half of which $\beta = 0.7$, are shown on the right in Fig. 5. The dissipation parameter $\alpha = 0.05$ remained constant throughout the whole rod. Distortions of the shape of the waves and the non-monotonic form of the amplitude variation can be seen. For comparison, the change in the stresses in a homogeneous rod, when $\beta = 0.7$ and α has the same value, is shown by the dashed curves.

The results obtained show that the proposed model enables one to give an adequate description of frequency-independent energy dissipation in structures and materials for a wide spectrum of dynamic processes.

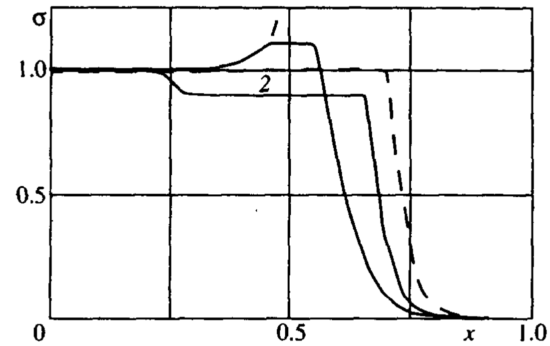

Fig. 4

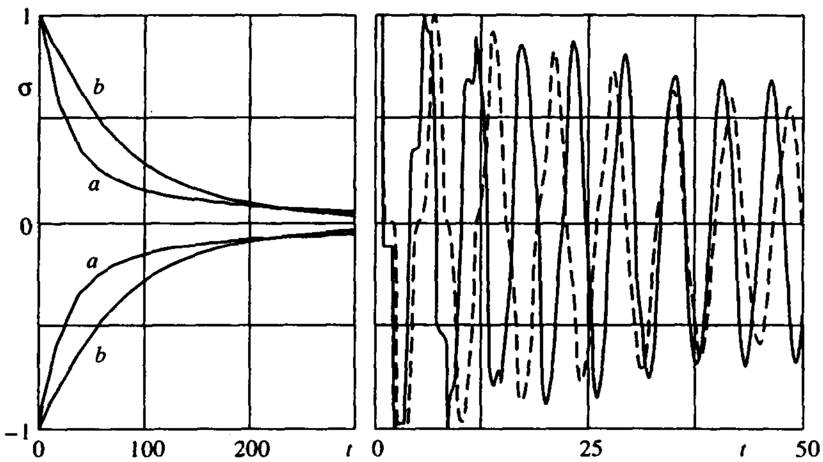

Fig. 5

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